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Self-Affine Fractal Clusters: Conceptual Questions and Numerical Results for Directed Percolation

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In this paper we address the question of the existence of a well defined, nontrivial fractal dimension D of self-affine clusters. In spite of the obvious relevance of such clusters to a wide range of phenomena, this problem is still open since the *different* published predictions for D have not been tested yet. An interesting aspect of the problem is that a nontrivial global dimension for clusters is in contrast with the trivial global dimension of self-affine functions. As a much studied example of self-affine structures, we investigate the infinite directed percolation cluster at the threshold. We measured D in $d=2$ dimensions by the box counting method. Using a correction to scaling analysis, we obtained $D=1.765(10)$. This result does not agree with any of the proposed relations, but it favors $D=1+(1-\sigma\nu_{||})/\sigma\nu_{\perp}$, where $\nu_{||}$ and $\nu_{\perp}$ are the correlation length exponents and σ is a Fisher exponent in the cluster scaling.

KEY WORDS: Fractal dimension; self-affinity; directed percolation.

1. INTRODUCTION

A realization of the importance of fractal geometry in describing a large variety of patterns occurring in nature has been one of the major developments of the last decade.[1-3] It seems that we are surrounded by objects which are scale invariant and have a nontrivial fractal dimension.

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The simplest kind of scale invariance is self-similarity, i.e., invariance with respect to homogeneous dilution or contraction. Recently, however, interest has increased in self-affine structures which are characterized by anisotropic scaling. The record of a one-dimensional random walker (the displacement vs. time function) or a rough surface are typical examples. By now much is known about this kind of self-affine fractal. Our paper deals with another category of self-affine fractals whose global dimension is not trivial. Considerably less is known about the fractal aspects of these objects. Obvious questions of the following nature still remain: do such clusters have a well-defined fractal dimension? If yes, how is its value related to the exponents describing the anisotropic scaling?

A variety of growth models leading to self-affine aggregates have been studied in the past (including directed animals and diffusion-limited aggregates in cylindrical geometry[5]). However, perhaps the simplest growth model resulting in branching self-affine clusters is directed percolation.[6–9] Directed percolation (DP) has much to recommend it as a model for study. It is the primary example of nonuniversal behavior in a percolation system. DP has many potential applications, including transport in a strong external field,[10] crack propagation,[11] and epidemics with a bias.[12] In addition, interesting mappings between directed percolation and various theoretical approaches have been established.[6]

In directed bond percolation, as in ordinary (undirected) percolation, bonds are occupied with a fixed probability p. In DP the bonds can be represented as arrows oriented according to an external direction and percolation against the arrows is forbidden. The special direction can often be considered as time, while directions orthogonal to it have spatial properties. There is a critical point p_c in this system in dimensions $d \geqslant 2$ so that for $p > p_c$ the probability of an infinite cluster is nonzero.

A cluster in DP is defined as the set of sites reachable from the origin via occupied directed paths. Typical clusters for $p \neq p_c$ are anisotropic and they are characterized by two different correlation lengths: $\xi_{||}$ (parallel to the time direction) and $\xi_{\perp}$ (perpendicular to it). As p approaches the critical point, the two correlation lengths diverge with different exponents:

$$\xi_{||} \sim |p - p_c|^{-\nu_{||}}, \qquad \xi_{\perp} \sim |p - p_c|^{-\nu_{\perp}} \tag{1}$$

The Markovian character of DP, i.e., the fact that the properties for time $t' > t_0$ do not depend on $t < t_0$ but on the configuration at $t = t_0$, makes the use of approximate methods easy and effective. However, no analytic solution of the critical behavior is known. Recently a number of results have been obtained for DP regarding, for example, the supercritical correlation length exponent in three dimensions,[13,14] the spectrum of the

transfer matrix,[15] the relation to self-organized criticality,[16,17] and violation of hyperscaling for DP.[18] The extension of the Kasteleyn–Fortuin formalism to DP makes similarities and differences between DP and ordinary percolation apparent.[19] In spite of all the recent attention, the fractal dimension of the DP infinite cluster at p_c has not yet been calculated directly.

In the present paper we first give a short discussion about the fractal dimension of self-affine structures and of the infinite DP cluster in particular (Section 2). In Section 3 the method of calculation and the algorithm are described. The results are presented in Section 4. We conclude the paper with a short summary.

2. SELF-AFFINE FRACTALS AND SCALING

In general, for structures which are invariant under a certain linear coordinate transformation, all elements of the corresponding transformation matrix could be different from zero. However, we restrict ourselves to cases where the objects are embedded into two-dimensional Euclidean space and the matrix is diagonal with different elements. The matrix then describes an affine transformation and objects invariant under such a transformation are called self-affine.[4]

The simplest such object is the graph of a power-law function. In order to arrive at a fractal one has to make this object "fuzzy" as in the Mandelbrot–Weierstrass function.[4] A random version is the record of displacement F vs. time t of a particle undergoing Brownian motion. The function $F(t)$ has the scaling property $F(t) \sim b^{-H}F(bt)$, at least in a statistical sense. (For Brownian motion $H = 1/2$.) Such a function is characterized by a nontrivial local fractal dimension D_l. If the graph of the function is covered by boxes of linear size ε, the number of boxes N_ε changes as $N_\varepsilon \sim \varepsilon^{-D_l}$ with $D_l = 2 - H$ for $\varepsilon \to 0$.

The global fractal dimension D is defined by introducing a lower cutoff in ε and counting the number N_ε of boxes covering the object so that larger and larger objects are considered while the cutoff is kept fixed. For self-affine functions of a scalar variable like the time the global dimension is trivially unity. As an illustration, let us consider the record of a discrete-time, discrete-step one-dimensional random walk. If it is plotted on a square lattice, then the global dimension is unity (see, e.g., p. 288 of ref. 2), since a square box of linear size ε can cover the random walk over ε time steps. Moreover, the equal units of the square lattice in time and space directions lead to the loss of the local fractal properties.[4] Of course, self-affine functions of higher-dimensional arguments can also be considered.

In order to obtain self-affine structures with nontrivial global fractal

properties one has to go beyond the self-affine functions. In most cases the direct product of two fractals of different fractal dimensions D_1 and D_2 leads to a self-affine object.[3] If we consider two fractals embedded into one-dimensional space, the direct product will be embedded into two dimensions and its x and y coordinates will be respectively the coordinates of the initial fractal sets. According to the addition rule,[1-3] the global fractal dimension of the direct product is then

$$D = D_1 + D_2 \tag{2}$$

If we cut such a cluster by straight lines parallel to the composing directions, we obtain the original fractal sets. It is known from percolation theory[6,20] that on a line cut along the parallel axis the infinite DP cluster $p = p_c$ has a fractal dimension

$$D_{\|} = 1 - \beta/\nu_{\|} \tag{3a}$$

For the cut in the perpendicular direction we have in d dimensions

$$D_{\perp} = d - 1 - \beta/\nu_{\perp} \tag{3b}$$

In (3), β is the exponent of the order parameter, i.e., the weight of the infinite cluster. If the infinite DP cluster can be considered as a self-affine fractal resulting from a direct product of fractals with dimensions $D_{\|}$ and $D_{\perp}$, then, according to (2), we would have

$$D = D_{\perp} + D_{\|} = d - \beta\left(\frac{1}{\nu_{\|}} + \frac{1}{\nu_{\perp}}\right) \tag{4}$$

Deterministic growth with an anisotropic seed can also result in a self-affine fractal. An example is the squeezed[21] Vicsek snowflake.[22] A possible way to calculate the fractal dimension of such an object is provided by the following argument. Let us suppose that in one direction the object grows as n^k and in the other as m^k while the number of particles grows as s^k. An estimate of the volume is given by $(nm)^k$. Thus, a characteristic length can be defined as $(nm)^{k/2}$. Using this rough argument, we arrive at

$$D = \frac{2 \log s}{\log n + \log m} \tag{5}$$

Such reasoning can be applied whenever the scaling of the width, the length, and the number of particles in the cluster is known.

This is the case for the infinite DP cluster at threshold. It can be obtained as the limiting object of finite clusters as p_c is approached from

below. Then the size of the typical cluster scales as $s \sim (p_c - p)^{-1/\sigma}$. The Fisher exponent σ appears in the scaling function f of the cluster numbers $n_s \sim s^{-\tau} f(s^{\sigma}(p - p_c))$ with τ being the other Fisher exponent. The width of the typical clusters is given by $\xi_{\perp}$, the length by $\xi_{||}$. Thus, the argument leading to (5) suggests

$$D = \frac{2}{\sigma(\nu_{||} + \nu_{\perp})} \tag{6}$$

or, in d dimensions,[6] $D = d/[\sigma(\nu_{||} + (d-1)\,\nu_{\perp})]$.

One can, however, arrive at a different formula by using the following argument.[5] For ordinary percolation there is only one characteristic length ξ in the system. The ratio s/ξ^D approaches a constant for $p \to p_c$: the infinite cluster of ordinary percolation at threshold is a self-similar fractal. For self-affine fractals a correction due to the two different characteristic lengths should be taken into account. For DP we assume this correction to have the form $s/\xi_{\perp}^D \sim \xi_{||}/\xi_{\perp}$, which expresses the fact that $\xi_{\perp} \ll \xi_{||}$ and leads to

$$D = 1 + \frac{1/\sigma - \nu_{||}}{\nu_{\perp}} \tag{7}$$

Assuming, on the other hand, that the cuts not in the parallel direction are generic, i.e., they have $D_{\perp}$ irrespective of the angle (P. Grassberger, private communication), one can use the formula[1] $D = 1 + D_{\perp}$, leading to

$$D = d - \frac{\beta}{\nu_{\perp}} \tag{8}$$

Formulas (7) and (8) are equivalent, since $1/\sigma = \nu_{||} + (d-1)\,\nu_{\perp} - \beta$. In the following we shall restrict ourselves to two dimensions.

3. THE ALGORITHM

The clusters which we study are constructed as follows. Consider a square lattice which is rotated 45 deg (so that all bonds close 45 deg with a horizontal line). In this way the lattice is made up of horizontal rows of sites at the vertices of double bonds ($\wedge$) and a directedness can be defined by associating with each bond an arrow pointing "down." One of the sites is chosen to be the origin. The two bonds connected to the origin from below are considered for occupation with probability p. This means that for each bond a random number r is generated and if $r < p$, the bond is

Fig. 1. Two-dimensional directed bond percolation cluster at p_c after 3072 time steps.

occupied. In the following step all bonds connected to the previous occupied bonds from below are occupied with probability p, and so on.

For p smaller than the critical p_c the growing cluster dies out, in most cases, after a few steps. If $p > p_c$, the cluster almost always grows forever and has a finite density and overall conical shape with a well-defined p-dependent angle going to zero as $p \to p_c$. At p_c the situation is qualitatively different in that, while many clusters may die out before becoming very large, in a reasonable fraction of attempts one can obtain extremely extended clusters which possess self-affine fractal properties.

The critical parameters of directed bond percolation on the square lattice are known to high accuracy[7]: $p_c = 0.644701(1)$, $\nu_{||} = 1.7339(3)$, $\nu_\perp = 1.0969(3)$, $\beta = 0.277(2)$, and $\sigma = 0.392(3)$. The length and width scale as $\xi_{||} \sim \xi_\perp^\theta$, where the anisotropy exponent $\theta = \nu_{||}/\nu_\perp = 1.581$.

Clusters were grown from a seed at p_c up to lengths $T = 2^k$ with k between 12 and 15. Typically, 20–30% of clusters reached the desired length. A typical directed percolation cluster extending to 3000 time steps is presented in Fig. 1, perhaps the first time such an extensive directed percolation cluster is depicted in the literature.

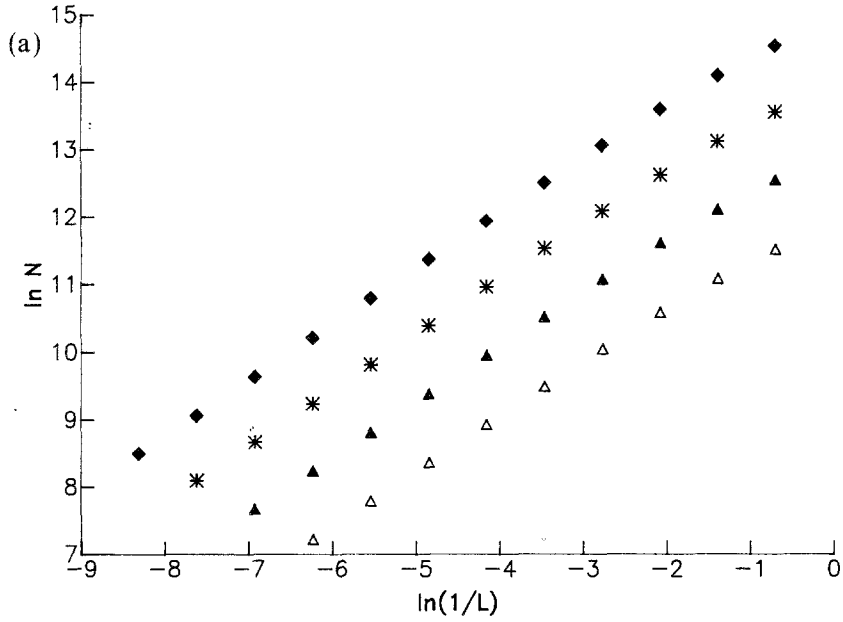
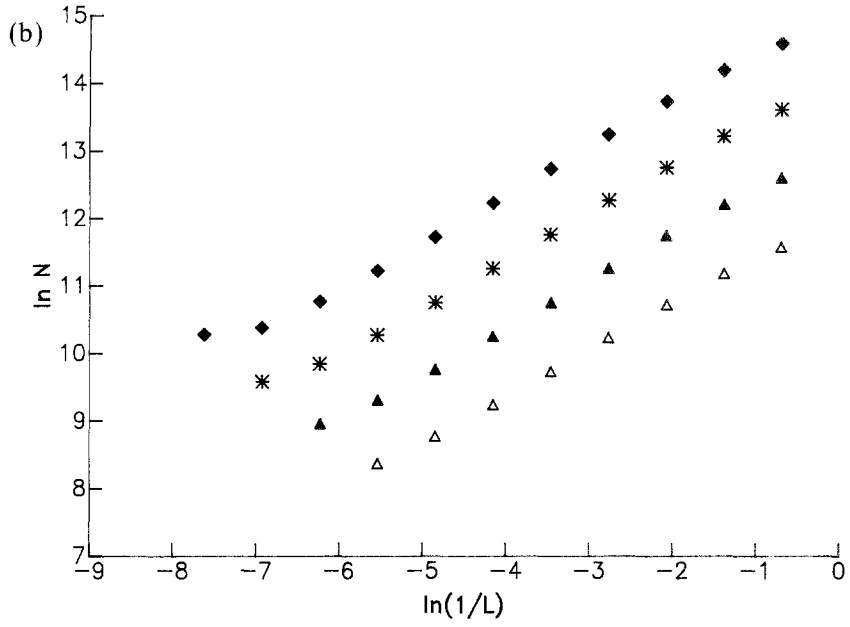


Fig. 2. Number of boxes required to cover all the cluster sites as a function of inverse box dimension L. Cluster length $T = 32{,}768$ (◆), 16,384 (★), 8192 (▲), 4096 (△). (a) Strip boxes along the parallel direction, (b) strip boxes along the perpendicular direction, (c) square boxes.



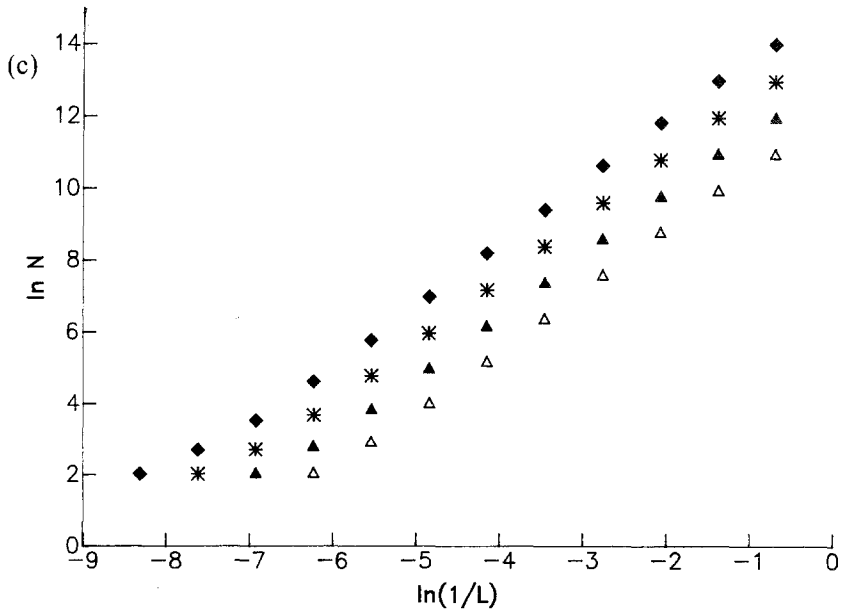


Fig. 2. (*Continued*)

The algorithm used one bit per site, so that 64 sites could be generated simultaneously on the Cray YMP. In order to further facilitate bit-coding, an array of constant width was used even though, in the initial stages, most of the bonds would be unoccupied. Array widths of between $T/4$ and $T/8$ were found to suitably contain the clusters considered here. With a vectorized algorithm employing multi-spin-coding as well as line-by-line updating and counting, it was possible to grow and analyze a system of length $T = 32{,}768$ and width $T/8$ using 15 sec of CPU time on the Cray YMP. This includes the full box counting analysis on the cluster.

The method used for determination of fractal dimension was box counting,[3] which counts the number of boxes N required to cover all of the sites. Boxes of size $L = 2$ to $L = T/8$ (in most cases) were used on the second half of those clusters that reached the required size. The counting proceeded concurrently with the growth so that it was necessary to store the data from just two rows at any given time. Three different calculations were performed so as to obtain the fractal dimension of a cut or strip along the parallel direction, of a cut along the perpendicular direction, as well as the two-dimensional fractal dimension.

For the perpendicular case, linear boxes were applied to a cut or strip along the perpendicular direction. Rather than consider just a single cut across the cluster, however, we made a cut each time a new row was

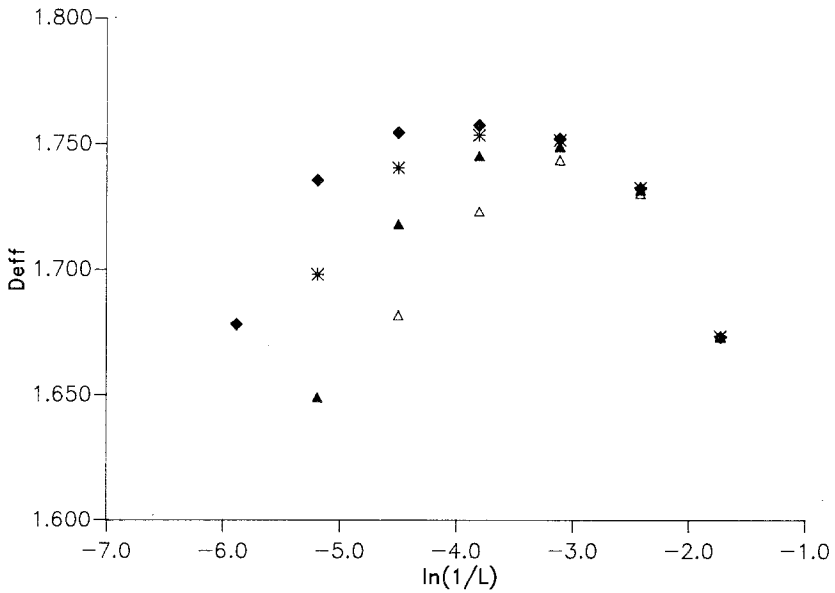


Fig. 3. Interval slopes of ln N vs. ln$(1/L)$ for two dimensions (from Fig. 2c). Cluster length $T = 32{,}768$ (◆), 16,384 (★), 8192 (▲), 4096 (△).

generated. For the parallel case, the cuts were oriented along the parallel direction so that successive boxes of size $L=2$ were counted at every second time step. If a given box were to contain an occupied site, then the box of size $L=4$ in which the smaller box was situated would also be occupied. In this way, boxes of size $L=4$ (8, 16, 32,...) needed to be considered only every 4 (8, 16, 32,...) time steps during the cluster growth. This method was also applied in the two-dimensional case using square boxes of dimension L.

4. RESULTS

The results of several hundred (in the case of size 32,768) to several thousand (in the case of size 4096) clusters were averaged. The results of box counting for various cluster lengths are presented in Fig. 2, which is a log-log plot of N vs. inverse box size ($1/L$). Finite-size effects due to system size and box size are visible as curvature at the ends. The slope of the central straight-line region corresponds to the effective fractal dimension D_{eff} for the finite system. A more detailed picture emerges from looking at the slopes between successive data points. Figure 3 presents the two-dimensional data and clearly shows an increase with system size. Similar results were obtained for the data of Figs. 2a and 2b. It is possible to infer the

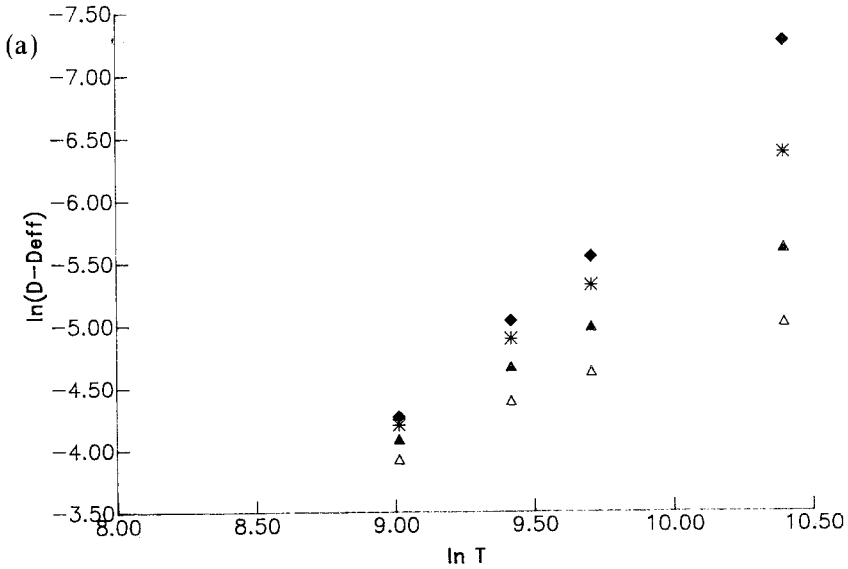
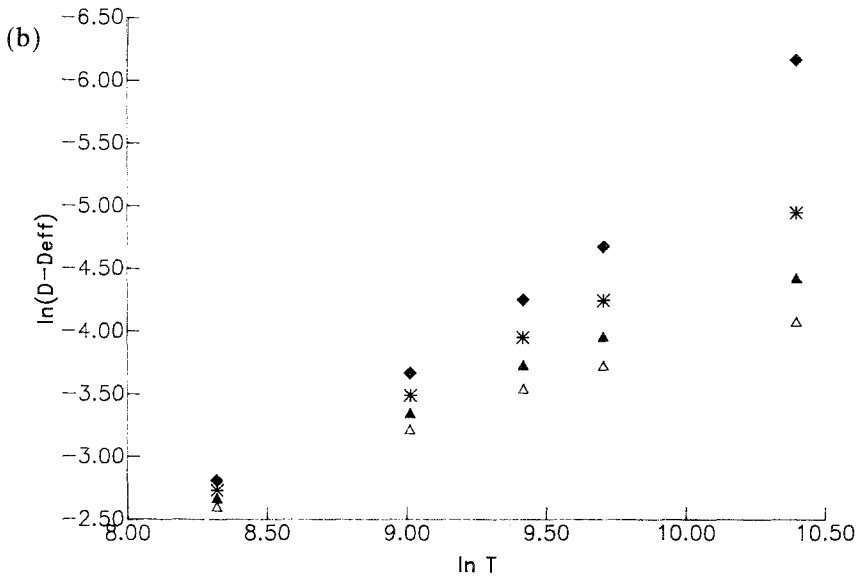


Fig. 4. $D-D_{\text{eff}}$ vs T. (a) Parallel direction, $D_{\parallel}=0.832$ (◆), 0.833 (★), 0.835 (▲), 0.838 (△); (b) perpendicular direction, $D_{\perp}=0.735$ (◆), 0.740 (★), 0.745 (▲), 0.750 (△); (c) two dimensions, $D=1.760$ (◆), 1.762 (★), 1.765 (▲), 1.770 (△).



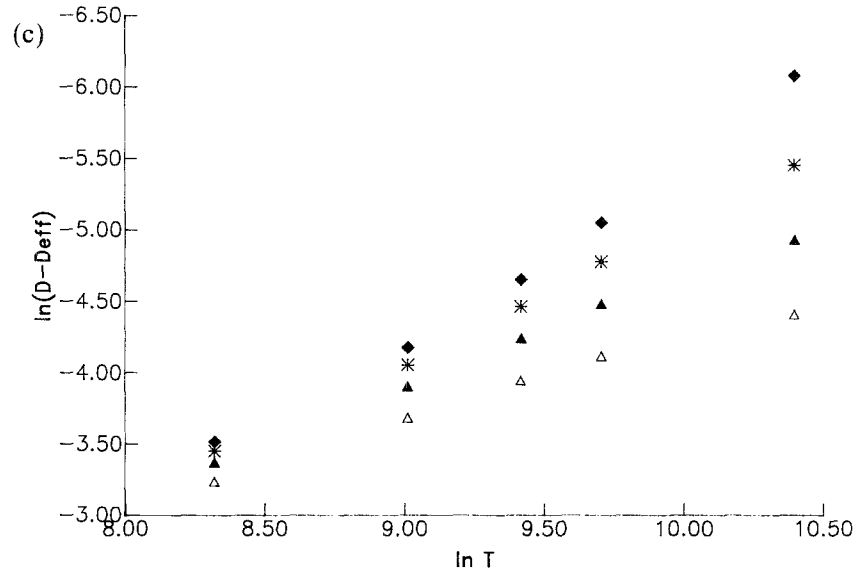


Fig. 4. (*Continued*)

Table I. Comparison of Proposed and Calculated Values of the Fractal Dimension D

Equation	D
(4)	1.585(5)
(6)	1.804(13)
(7) or (8)	1.747(3)
Our result	1.765(10)

fractal dimension for the case of an infinite cluster by assuming scaling with system length T of the form

$$D_{\text{eff}} = D(1 + cT^{-\kappa}) \tag{9}$$

where D_{eff} refers to systems of finite size and D refers to the infinite system. Using several trial values for D and corresponding interval slopes as described in Fig. 3, for example, as the D_{eff} values, we obtained log-log plots of $(D - D_{\text{eff}})$ vs. T (Fig. 4). The true value of D in each case would correspond asymptotically to a straight-line graph. We estimate $D_{||} = 0.833(3)$, $D_{\perp} = 0.740(5)$, and for the fractal dimension of the whole cluster $D = 1.765(5)$. The correction to scaling exponent κ is determined less precisely as $\kappa = 0.8(2)$. The value of the correction to scaling exponent is comparable to other estimates.[18]

The results were obtained by using only the critical value p_c as input. However, if we compare our estimates for $D_{||}$ and $D_{\perp}$ with the best available data from the literature,[7] we realize the possibility of uncontrolled systematic errors. While for $D_{\perp}$ our error bar overlaps with that of the best estimate [0.747(2)], for $D_{||}$ we should increase our error bar by a factor of two in order to overlap with the best estimate 0.840(2). This implies that the suggested error in D of 0.005 might also be larger by a factor of about two.

In Table I we summarize the proposed values for the fractal dimension of critical directed percolation clusters in two dimensions together with our numerical estimate with the doubled error bar. It is clear that the present calculation favors (7).

5. CONCLUSIONS

We have discussed various conceptual questions arising in the context of determining the global fractal dimension of self-affine clusters. The complexity of the situation is demonstrated by the fact that different heuristic arguments lead to expressions for D which are inconsistent with each other.

Apparently, at present there is no theoretical basis to single out which equation for D is the correct one.

Therefore, using simulations, we have obtained accurate numerical estimates for D, $D_{||}$, and $D_{\perp}$ for the case of two-dimensional directed percolation clusters at the threshold and compared our estimate with those available in the literature. Our calculations have shown that only very large-scale simulations can be conclusive enough for this purpose. None of the heuristically derived formulas agree with our result. However, it is clear that the difference from (7) [or (8)] is so small that these formulas are favored over the other proposed relations.

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